

# New Moduli for Banach Spaces

Grigory Ivanov<sup>1</sup> and Horst Martin<sup>2</sup>

**Abstract.** Modifying the moduli of supporting convexity and supporting smoothness, we introduce new moduli for Banach spaces which occur, e.g., as lengths of catheti of right-angled triangles (defined via so-called quasi-orthogonality). These triangles have two boundary points of the unit ball of a Banach space as endpoints of their hypotenuse, and their third vertex lies in a supporting hyperplane of one of the two other vertices. Among other things it is our goal to quantify via such triangles the local deviation of the unit sphere from its supporting hyperplanes. We prove respective Day-Nordlander type results, involving generalizations of the modulus of convexity and the modulus of Banaś.

Mathematics Subject Classification (2010): 46B07, 46B20, 52A10, 52A20, 52A21

Keywords: Birkhoff-James orthogonality, Day-Nordlander type results, Milman modulus, modulus of (supporting) convexity, modulus of (supporting) smoothness, quasi-orthogonality

## 1. INTRODUCTION

The modulus of convexity (going back to [10]) and the modulus of smoothness (defined in [12]) are well known classical constants from Banach space theory. For these two notions various interesting applications were found, and a large variety of natural refinements, generalizations, and modifications of them created an impressive bunch of interesting results and problems; see, e.g., [23], [22], [26], [5], [8], and [20], to cite only references close to our discussion here. Inspired by [5], two further constants in this direction were introduced and investigated in [20], namely the modulus of supporting convexity and the modulus of supporting smoothness. These moduli suitably quantify the local deviation of the boundary of the unit ball of a real Banach space from its supporting hyperplanes near to arbitrarily chosen touching points. Using the concept of right-angled triangles in terms of so-called quasi-orthogonality (which is closely related to the concept of Birkhoff-James orthogonality), we modify and complete the framework of moduli defined in [10], [5], and [20] by introducing and studying new related constants. These occur as lengths of catheti of such triangles, whose hypotenuse connects two boundary points of the unit ball and whose third vertex lies in the related supporting hyperplane. We prove Day-Nordlander type results referring to these moduli, yielding even generalizations of the constants introduced in [10], [5], and [20]. Respective results on Hilbert spaces are obtained, too. At the end we discuss some conjectures and questions which refer to further related inequalities between such moduli (for general Banach spaces, but also for Hilbert spaces), possible characterizations of inner product spaces, and Milman's moduli.

<sup>1</sup> DCG, FSB, Ecole Polytechnique Fédérale de Lausanne, Route Cantonale, 1015 Lausanne, Switzerland.

Department of Higher Mathematics, Moscow Institute of Physics and Technology, Institutskii pereulok 9, Dolgoprudny, Moscow region, 141700, Russia.

grimivanov@gmail.com

Research partially supported by Swiss National Science Foundation grants 200020-165977 and 200021-162884

Supported by Russian Foundation for Basic Research, project 16-01-00259.

<sup>2</sup>Faculty of Mathematics, TU Chemnitz, 09107 Chemnitz, Germany

The paper is organized as follows: After presenting our notation and basic definitions in Section 2, we clarify the geometric position of the mentioned right-angled triangles close to a point of the unit sphere of a Banach space and its corresponding supporting hyperplane. This yields a clear geometric presentation of the new moduli, but also of further moduli already discussed in the literature. In Section 4 we particularly study properties of the catheti of these triangles, yielding also the announced results of Day-Nordlander type and results on Hilbert spaces. In a similar way, we study properties of the hypotenuses in Section 5, obtaining again Day-Nordlander type results and further new geometric inequalities. In Section 6 our notions and results are put into a more general framework, connected with concepts like monotone operators, dual mappings of unit spheres and their monotonicity. And in Section 7 some open questions and conjectures on the topics shortly described above are collected.

## 2. NOTATION AND BASIC DEFINITIONS

In the sequel we shall need the following notation. Let  $X$  be a *real Banach space*, and  $X^*$  be its *conjugate space*. We use  $H$  to denote a *Hilbert space*. For a set  $A \subset X$  we denote by  $\partial A$  and  $\text{int } A$  the *boundary* and the *interior* of  $A$ , respectively. We use  $\langle p, x \rangle$  to denote the *value of a functional*  $p \in X^*$  at a vector  $x \in X$ . For  $R > 0$  and  $c \in X$  we denote by  $\mathfrak{B}_R(c)$  the *closed ball with center*  $c$  and *radius*  $R$ , and by  $\mathfrak{B}_R^*(c)$  the respective *ball in the conjugate space*. Thus,  $\partial \mathfrak{B}_1(o)$  denotes the *unit sphere* of  $X$ . By definition, we put  $J_1(x) = \{p \in \partial \mathfrak{B}_1^*(o) : \langle p, x \rangle = \|x\|\}$ .

We will use the notation  $xy$  for the *segment* with the (distinct) endpoints  $x$  and  $y$ , for the *line* passing through these points, for (oriented) *arcs* from  $\partial \mathfrak{B}_R(c)$ , as well as for the *vector* from  $x$  to  $y$  (the respective meaning will always be clear by the context). Further on, abbreviations like  $abc$  and  $abcd$  are used for *triangles* and *4-gons* as convex hulls of these three or four points.

We say that  $y$  is *quasi-orthogonal* to the vector  $x \in X \setminus \{o\}$  and write  $y \perp x$  if there exists a functional  $p \in J_1(x)$  such that  $\langle p, y \rangle = 0$ . Note that the following conditions are equivalent:

- $y$  is quasi-orthogonal to  $x$ ;
- for any  $\lambda \in \mathbb{R}$  the vector  $x + \lambda y$  lies in the supporting hyperplane to the ball  $\mathfrak{B}_{\|x\|}(o)$  at  $x$ ;
- for any  $\lambda \in \mathbb{R}$  the inequality  $\|x + \lambda y\| \geq \|x\|$  holds;
- $x$  is orthogonal to  $y$  in the sense of Birkhoff–James (see [14], Ch. 2, §1, and [3]).

Let

$$\delta_X(\varepsilon) := \inf \left\{ 1 - \frac{\|x + y\|}{2} : x, y \in \mathfrak{B}_1(o), \|x - y\| \geq \varepsilon \right\}$$

and

$$\rho_X(\tau) := \sup \left\{ \frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau \right\}.$$

The functions  $\delta_X(\cdot) : [0, 2] \rightarrow [0, 1]$  and  $\rho_X(\cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  are referred to as the *moduli of convexity and smoothness of*  $X$ , respectively.

In [5] J. Banaś defined and studied some new modulus of smoothness. Namely, he defined

$$\delta_X^+(\varepsilon) = \sup \left\{ 1 - \frac{\|x + y\|}{2} : x, y \in \mathfrak{B}_1(o), \|x - y\| \leq \varepsilon \right\}, \quad \varepsilon \in [0, 2].$$

Let  $f$  and  $g$  be two non-negative functions, each of them defined on a segment  $[0, \varepsilon]$ . We shall say that  $f$  and  $g$  are *equivalent at zero*, denoted by  $f(t) \asymp g(t)$  as  $t \rightarrow 0$ , if there exist positive constants  $a, b, c, d, e$  such that  $af(bt) \leq g(t) \leq cf(dt)$  for  $t \in [0, e]$ .

### 3. RIGHT-ANGLED TRIANGLES

We will say that a triangle is *right-angled* if one of its legs is quasi-orthogonal to the other one. (Note that there are completely different ways to define right-angled triangles in normed planes, see also [2].) In a Hilbert space this notion coincides with the common, well-known definition of a right-angled triangle.

**Remark 1.** *In a non-smooth convex Banach space one leg of a triangle can be quasi-orthogonal to the two others.*

For a given right-angled triangle  $abc$ , where  $ac \perp bc$ , we will say that the legs  $ac, bc$  are the *catheti*, and  $ab$  the *hypotenuse*, of this triangle.

For convenience we draw a simple figure (see Fig. 1) and introduce related new moduli by explicit geometric construction. Let  $x, y \in \partial\mathfrak{B}_1(o)$  be such that  $y \perp x$ . Let  $\varepsilon \in (0, 1]$ ,  $y_1 = x + \varepsilon y$ . Denote by  $z$  a point from the unit sphere such that  $zy_1 \parallel ox$  and  $zy_1 \cap \mathfrak{B}_1(o) = \{z\}$ . Let  $\{d\} = oy_1 \cap \partial\mathfrak{B}_1(o)$ . Write  $y_2$  for the projection of the point  $d$  onto the line  $\{x + \tau y : \tau \in \mathbb{R}\}$  (in the non-strictly convex case we choose  $y_2$  such that  $dy_2 \parallel ox$ ). Let  $p \in J_1(x)$  be such that  $\langle p, y \rangle = 0$ , i.e., the line  $\{x + \tau y : \tau \in \mathbb{R}\}$  lies in the supporting hyperplane  $l = \{a \in X : \langle p, a \rangle = 1\}$  of the unit ball at the point  $x$ . Then  $\|zy_1\| = \langle p, x - z \rangle$ .

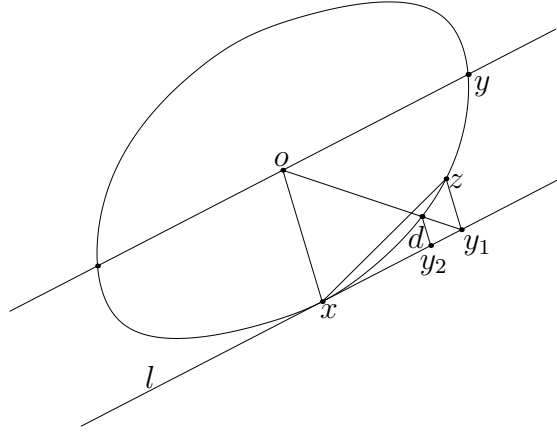


FIGURE 1. Right-angled triangles and the unit sphere

Consider the right-angled triangle  $oxy_1$  (Fig. 1). In a Hilbert space we have  $\|oy_1\| = \sqrt{1 + \varepsilon^2}$ , but in an arbitrary Banach space the length of the hypotenuse  $oy_1$  can vary. So we introduce moduli that describe the minimal and the maximal length of the hypotenuse in a right-angled triangle in a Banach space. More precisely, we write

$$\zeta_X^-(\varepsilon) := \inf \{ \|x + \varepsilon y\| : x, y \in \partial\mathfrak{B}_1(o), y \perp x \}$$

and

$$\zeta_X^+(\varepsilon) := \sup \{ \|x + \varepsilon y\| : x, y \in \partial\mathfrak{B}_1(o), y \perp x \},$$

where  $\varepsilon$  is an arbitrary positive real number.

In other words,  $\zeta_X^-(\cdot) - 1$  and  $\zeta_X^+(\cdot) - 1$  describe extrema of the *deviation of a point in a supporting hyperplane from the unit ball*.

On the other hand, the length of the segment  $zy_1$  is the *deviation of a point at the unit sphere from the corresponding supporting hyperplane*, and at the same time it is a cathetus in the triangle  $xzy_1$ .

Let  $x, y \in \partial\mathfrak{B}_1(o)$  be such that  $y \lrcorner x$ . By definition, put

$$\lambda_X(x, y, \varepsilon) := \min \{ \lambda \in \mathbb{R} : \|x + \varepsilon y - \lambda x\| = 1 \}$$

for any  $\varepsilon \in [0, 1]$ . In the notation of Fig. 1 we have  $\lambda_X(x, y, \varepsilon) = \|zy_1\|$ . The minimal and the maximal value of  $\lambda_X(x, y, \varepsilon)$  characterize the deviation of the unit sphere from an arbitrary supporting hyperplane. Let us introduce now further moduli.

Define the *modulus of supporting convexity* by

$$\lambda_X^-(\varepsilon) = \inf \{ \lambda_X(x, y, \varepsilon) : x, y \in \mathfrak{B}_1(o), y \lrcorner x \},$$

and the *modulus of supporting smoothness* by

$$\lambda_X^+(\varepsilon) = \sup \{ \lambda_X(x, y, \varepsilon) : x, y \in \mathfrak{B}_1(o), y \lrcorner x \}.$$

The notions of moduli of supporting convexity and supporting smoothness were introduced and studied in [20]. These moduli are very convenient for solving problems concerning the local behaviour of the unit ball compared with that of corresponding supporting hyperplanes. We will use some of their properties in this paper.

In [20] the following inequalities were proved:

$$(1) \quad \rho_X\left(\frac{\varepsilon}{2}\right) \leq \lambda_X^+(\varepsilon) \leq \rho_X(2\varepsilon), \quad \varepsilon \in \left[0, \frac{1}{2}\right],$$

$$(2) \quad \delta_X(\varepsilon) \leq \lambda_X^-(\varepsilon) \leq \delta_X(2\varepsilon), \quad \varepsilon \in [0, 1],$$

and

$$(3) \quad 0 \leq \lambda_X^-(\varepsilon) \leq \lambda_X^+(\varepsilon) \leq \varepsilon.$$

In addition, also a Day-Nordlander type result, referring to these moduli, was proved in [20]:

$$\lambda_X^-(\varepsilon) \leq \lambda_H^-(\varepsilon) = 1 - \sqrt{1 - \varepsilon^2} = \lambda_H^+(\varepsilon) \leq \lambda_X^+(\varepsilon) \quad \forall \varepsilon \in [0, 1].$$

In some sense, moduli of supporting convexity and supporting smoothness are estimates of a possible value referring to tangents in a Banach space (we fix the length of one of the catheti and calculate then the minimal and maximal length of the correspondingly other cathetus, which is quasi-orthogonal to the first one).

**Remark 2.** By convexity of the unit ball we have that, for arbitrary  $x, y \in \mathfrak{B}_1(o)$  such that  $y \lrcorner x$ , the function  $\lambda_X(x, y, \cdot)$  is a convex function on the interval  $[0, 1]$ .

But what can one say about the length of the segment  $zy_1$  with fixed norm  $\|zx\|$  (in the notation of the Fig. 1)?

Let us introduce the following new moduli of a Banach space:

$$(4) \quad \varphi_X^-(\varepsilon) = \inf \{ \langle p, x - z \rangle : x, z \in \partial\mathfrak{B}_1(o), \|x - z\| \geq \varepsilon, p \in J_1(x) \}$$

and

$$(5) \quad \varphi_X^+(\varepsilon) = \sup \{ \langle p, x - z \rangle : x, z \in \partial\mathfrak{B}_1(o), \|x - z\| \leq \varepsilon, p \in J_1(x) \}$$

for  $\varepsilon \in [0, 2]$ .

**Remark 3.** Due to the convexity of the unit ball we can substitute inequalities in the definitions of  $\varphi_X^-(\cdot)$  and  $\varphi_X^+(\cdot)$  to equalities (i.e.,  $\|x - y\| \geq \varepsilon$  and  $\|x - y\| \leq \varepsilon$  to be  $\|x - y\| = \varepsilon$ ).

#### 4. PROPERTIES OF THE CATHETI

**Lemma 1.** In the notation of Fig. 1, we have  $2\|y_1x\| \geq \|xz\|$ .

**Proof.**

By the triangle inequality, it suffices to show that  $\|y_1x\| \geq \|zy_1\|$ . Let the line  $\ell_y$  be parallel to  $ox$  with  $y \in \ell_y$ . By construction, we have that the points  $x, y_1, z, o, y$  and the line  $\ell_y$  lie in the same plane – the linear span of the vectors  $x$  and  $y$ . So the lines  $\ell_y$  and  $xy_1$  intersect, and by  $c$  we denote their intersection point. Note that  $oycx$  is a parallelogram and  $\|yc\| = 1$ ; the segment  $yx$  belongs to the unit ball and does not intersect the interior of the segment  $zy_1$ . Let  $\{z'\} = zy_1 \cap yx$ . By similarity, we have

$$\|zy_1\| \leq \|y_1z'\| = \frac{\|xy_1\|}{\|xc\|} \|yc\| = \|xy_1\|.$$

□

It is worth noticing that under the conditions of Lemma 1 we have that  $y_1$  is a projection along

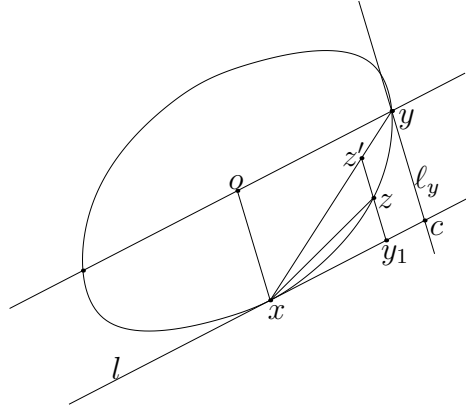


FIGURE 2. Under the conditions of Lemma 1 we have  $2\|xy_1\| \geq \|xz\|$ .

the vector  $ox$  of the point  $z$  on some supporting hyperplane of the unit ball at  $x$ . Moreover,  $y_1$  belongs to the metric projection of the point  $y$  on this hyperplane. In other words, Lemma 1 shows us that if one projects the segment  $xz$  along the vector  $ox$  onto the hyperplane which supports the unit ball at  $x$ , then the length of the segment decreases no more than by a factor of 2.

**Lemma 2.** Let  $X$  be an arbitrary Banach space. Then the following inequalities hold:

$$(6) \quad \lambda_X^-\left(\frac{\varepsilon}{2}\right) \leq \varphi_X^-(\varepsilon) \leq \lambda_X^-(2\varepsilon) \quad \text{and}$$

$$(7) \quad \lambda_X^+\left(\frac{\varepsilon}{2}\right) \leq \varphi_X^+(\varepsilon) \leq \lambda_X^+(2\varepsilon)$$

for  $\varepsilon \in [0, 1/2]$ .

**Proof.**

In the notation of Fig.1 we assume that for arbitrary  $x, y$  with  $y \perp x$  the equality  $\|zx\| = \varepsilon$  holds. Then  $\lambda_X(x, y, \|xy_1\|) = \|y_1z\|$ . Let  $p \in J_1(x)$  be such that  $\langle p, y \rangle = 0$ . Hence  $\|y_1z\| = \langle p, x - y \rangle$ . Since  $\|xy_1\| \leq \|y_1z\| + \|zx\| \leq 2\varepsilon$ , and taking into account Lemma 1, we get

$$\frac{\varepsilon}{2} \leq \|xy_1\| \leq 2\varepsilon \leq 1.$$

Due to this and by Remark 2, we have

$$\lambda_X\left(x, y, \frac{\varepsilon}{2}\right) \leq \langle p, x - y \rangle \leq \lambda_X(x, y, 2\varepsilon).$$

Taking infimum (supremum) on the right-hand side, left-hand side or in the middle part of the last inequality, we obtain (6) and (7).  $\square$

From Lemma 2 and the inequalities (2) and (1) we have the following corollary.

**Corollary 1.** *Let  $X$  be an arbitrary Banach space. Then  $\varphi_X^+(\varepsilon) \asymp \rho_X(\varepsilon)$  and  $\varphi_X^-(\varepsilon) \asymp \delta_X(\varepsilon)$  as  $\varepsilon \rightarrow 0$ , and for  $\varepsilon \in [0, \frac{1}{2}]$  the following inequalities hold:*

$$\rho_X\left(\frac{\varepsilon}{4}\right) \leq \varphi_X^+(\varepsilon) \leq \rho_X(4\varepsilon), \text{ and}$$

$$\delta_X(\varepsilon) \leq \varphi_X^-(\varepsilon) \leq \delta_X(4\varepsilon).$$

Now we will prove a Day-Nordlander type result for  $\varphi_X^-(\cdot)$  and  $\varphi_X^+(\cdot)$ . Let us suitably generalize the notion of modulus of convexity and the notion of Banaś modulus. Namely, let

$$\delta_X(\varepsilon, t) = \inf \left\{ 1 - \frac{\|tx + (1-t)y\|}{2} : x, y \in \partial \mathfrak{B}_1(o), \|x - y\| = \varepsilon \right\}$$

and

$$\delta_X^+(\varepsilon, t) = \sup \left\{ 1 - \frac{\|tx + (1-t)y\|}{2} : x, y \in \partial \mathfrak{B}_1(o), \|x - y\| = \varepsilon \right\},$$

respectively. Using the same method as in the classical paper [23], we get

**Lemma 3.** *Let  $X$  be an arbitrary Banach space. Then the following inequalities hold:*

$$(8) \quad \delta_X(\varepsilon, t) \leq \delta_H(\varepsilon, t) = 1 - \sqrt{1 - t(1-t)\varepsilon^2} = \delta_H^+(\varepsilon, t) \leq \delta_X^+(\varepsilon, t).$$

**Proof.**

Since the proof is almost the same as in [23], we present only a short sketch. Clearly, again it is sufficient to prove the lemma in the two-dimensional case.

If the two unit vectors  $x = (x_1, x_2)$  and  $y = (y_1, y_2)$  are rotated around the unit circle, while their difference  $x - y$  has constantly the norm  $\varepsilon$ , the endpoint of the vector  $tx + (1-t)y$  describes a curve  $\Gamma_t$ .

The following integral expresses the area of the region inside the curve described by the endpoint of the vector  $x - y$ , if this vector is laid off from a fixed point:

$$\int (y_1 - x_1) d(y_2 - x_2).$$

On the other hand, the mentioned curve is a homothet of the unit circle with ratio  $\varepsilon$ . Hence this integral equals  $\varepsilon^2 A$ , where  $A$  is the area of the unit ball ( $A = \int x_1 dx_2 = \int y_1 dy_2$ ). From this we have that

$$\int x_1 dy_2 + \int y_1 dx_2 = 2A - \varepsilon^2 A.$$

Now it is clear that the area of the region inside  $\Gamma_t$  equals

$$\int (tx_1 + (1-t)y_1) d(tx_2 + (1-t)y_2) = A(1-t(1-t)\varepsilon^2).$$

Hence continuity arguments imply that there exists a point  $z \in \Gamma_t$  with the norm  $\sqrt{1-t(1-t)\varepsilon^2}$ .  $\square$

**Theorem 1.** *Let  $X$  be an arbitrary Banach space. Then the following inequalities hold:*

$$(9) \quad \varphi_X^-(\varepsilon) \leq \varphi_H^-(\varepsilon) = \frac{\varepsilon^2}{2} = \varphi_H^+(\varepsilon) \leq \varphi_X^+(\varepsilon).$$

**Proof.**

It is sufficient to prove the theorem in the two-dimensional case. Let  $x \in \partial\mathfrak{B}_1(o)$  and  $p \in J_1(x)$ .

Assume that  $X$  is a uniformly smooth space. Notice that  $p$  is a Frechet derivative of the norm at the point  $x$ . Taking into account that  $\mathfrak{B}_1(o)$  is convex, for an arbitrary  $y$  we have

$$\langle p, x - y \rangle = \lim_{t \searrow 0} \frac{\|x\| - \|x + t(y - x)\|}{t} = \liminf_{t > 0} \frac{1 - \|x + t(y - x)\|}{t}.$$

Fix an arbitrary  $\gamma > 0$ . Since  $X$  is uniformly smooth, there exists a  $t_0 < \gamma$  such that for arbitrary  $x, y \in \partial\mathfrak{B}_1(o)$ ,  $\|y - x\| = \varepsilon$  and  $t \in (0, t_0)$  we have

$$(10) \quad \frac{1 - \|x + t(y - x)\|}{t} - \gamma \leq \langle p, x - y \rangle \leq \frac{1 - \|x + t(y - x)\|}{t}.$$

Taking the infimum (supremum) in the last line, we get

$$\begin{aligned} \frac{\delta_X(\varepsilon, t)}{t} - \gamma &\leq \varphi_X^-(\varepsilon) \leq \frac{\delta_X(\varepsilon, t)}{t} \\ \left( \frac{\delta_X^+(\varepsilon, t)}{t} - \gamma \leq \varphi_X^+(\varepsilon) \leq \frac{\delta_X^+(\varepsilon, t)}{t} \right). \end{aligned}$$

Passing to the limit as  $\gamma \rightarrow 0$ , we have

$$\begin{aligned} \varphi_X^-(\varepsilon) &= \lim_{t \rightarrow 0} \frac{\delta_X(\varepsilon, t)}{t} \leq \lim_{t \rightarrow 0} \frac{\delta_H(\varepsilon, t)}{t} = \frac{\varepsilon^2}{2} \\ \left( \varphi_X^+(\varepsilon) &= \lim_{t \rightarrow 0} \frac{\delta_X^+(\varepsilon, t)}{t} \geq \lim_{t \rightarrow 0} \frac{\delta_H^+(\varepsilon, t)}{t} = \frac{\varepsilon^2}{2} \right). \end{aligned}$$

Let us now consider the case of a non-smooth space  $X$ . Let  $SP$  be the set of all points of smoothness at the unit circle. We know that the unit circle is compact. Then there exists  $t_0 < \gamma$  such that for arbitrary  $x \in SP$ ,  $y \in \partial\mathfrak{B}_1(o)$ ,  $\|y - x\| = \varepsilon$  and  $t \in (0, t_0)$  we can write the inequality (10).

Moreover, the set  $\partial\mathfrak{B}_1(o) \setminus SP$  has measure zero. Thus, the infimum (supremum) of  $1 - \|x + t(y - x)\|$  taken over all  $x \in SP$  coincides with  $\delta_X(\varepsilon, t)$  ( $\delta_X^+(\varepsilon, t)$ ). So we have

$$\varphi_X^-(\varepsilon) \leq \limsup_{t \rightarrow 0} \frac{\delta_X(\varepsilon, t)}{t} \leq \frac{\varepsilon^2}{2}$$

and

$$\varphi_X^+(\varepsilon) \geq \liminf_{t \rightarrow 0} \frac{\delta_X^+(\varepsilon, t)}{t} \geq \frac{\varepsilon^2}{2}.$$

□

## 5. PROPERTIES OF THE HYPOTENUSE

**Lemma 4.** *Let  $X$  be an arbitrary Banach space. Then for  $\varepsilon \in [0, 1]$  the following inequalities hold:*

$$(11) \quad \lambda_X^-\left(\frac{\varepsilon}{1+\varepsilon}\right) \leq \zeta_X^-(\varepsilon) - 1 \leq \lambda_X^-(\varepsilon),$$

$$(12) \quad \lambda_X^+\left(\frac{\varepsilon}{1+\varepsilon}\right) \leq \zeta_X^+(\varepsilon) - 1 \leq \lambda_X^+(\varepsilon).$$

**Proof.**

From the triangle inequality we have that  $\|y_1 d\|$  equals the distance from the point  $y_1$  to the unit ball. Hence

$$(13) \quad \|y_1 d\| \leq \|y_1 z\| = \lambda_X(x, y, \varepsilon) \leq \varepsilon.$$

By similarity arguments and (13) we have

$$\|xy_2\| = \frac{\|od\|}{\|od\| + \|dy_1\|} \|xy_1\| = \frac{1}{1 + \|dy_1\|} \varepsilon \geq \frac{\varepsilon}{1 + \varepsilon}.$$

Then, by construction and by the convexity of the unit ball, we get the inequality

$$(14) \quad \|y_2 d\| = \lambda_X(x, y, \|xy_2\|) \geq \lambda_X\left(x, y, \frac{\varepsilon}{1 + \varepsilon}\right).$$

Since  $y_2$  is a projection of the point  $d$  onto the line  $\{x + \tau y : \tau \in \mathbb{R}\}$ , we have  $\|y_2 d\| \leq \|dy_1\|$ . Combining the previous inequality with (13) and (14), we obtain the inequalities

$$\lambda_X\left(x, y, \frac{\varepsilon}{1 + \varepsilon}\right) \leq \|dy_1\| \leq \lambda_X(x, y, \varepsilon).$$

Taking infimum (supremum) on the right-hand side, left-hand side or in the middle part of the last line, we obtain (11) and (12).

□

**Corollary 2.** *Let  $X$  be an arbitrary Banach space. Then  $\zeta_X^+(\varepsilon) - 1 \asymp \rho_X(\varepsilon)$  and  $\zeta_X^-(\varepsilon) - 1 \asymp \delta_X(\varepsilon)$  as  $\varepsilon \rightarrow 0$ , and the following inequalities hold:*

$$\rho_X\left(\frac{\varepsilon}{2(1+\varepsilon)}\right) \leq \zeta_X^+(\varepsilon) \leq \rho_X(2\varepsilon), \quad \varepsilon \in \left[0, \frac{1}{2}\right], \text{ and}$$



$$\delta_X\left(\frac{\varepsilon}{1+\varepsilon}\right) \leq \zeta_X^-(\varepsilon) \leq \delta_X(2\varepsilon), \quad \varepsilon \in [0, 1].$$

Now we will prove results of Day-Nordlander type for  $\zeta_X^-(\cdot)$  and  $\zeta_X^+(\cdot)$ .

Suppose we have an orientation  $\omega$  in  $\mathbb{R}^2$ . We will say that a curve  $C$  in the plane is a *good curve* if it is a closed rectifiable simple Jordan curve, which is enclosed by a star-shaped set  $S$  with center at the origin and continuous radial function.

**Lemma 5.** *Let  $C_1$  be a closed simple Jordan curve enclosing the convex set  $S_1$  with area  $A_1 > 0$  and  $0 \in \text{int } S_1$ . Let  $C_2$  be a good curve, which is enclosing an area of measure  $A_2$ . Then*

- (1) *we can parametrize  $C_i$  by a function  $f^i(\cdot) : [0, 1) \rightarrow C_i$  ( $i = 1, 2$ ) in such a way that*
  - (a)  *$f^2(\tau)$  is a direction vector of the supporting line of the set  $S_1$  at the point  $f^1(\tau)$  for all  $\tau \in [0, 1)$ ;*
  - (b)  *$[f^1(\tau), f^2(\tau)] = \omega$  for all  $\tau \in [0, 1)$ ;*
  - (c) *the functions  $f^i(\cdot)$  ( $i = 1, 2$ ) are angle-monotone;*
- (2) *the curve  $C_3 = \{f^1(\tau) + f^2(\tau) : \tau \in [0, 1)\}$  encloses an area of measure  $A_1 + A_2$ .*

**Proof.**

1) First of all, due to the continuity of the radial function of the curve  $C_2$  we can assume that  $C_2$  and  $C_1$  are coincident.

Let  $C_1$  be a smooth curve. Let  $f^1 : [0, 1) \rightarrow C_1$  be a parametrization given by clockwise rotation. Then at every point  $f^1(\tau)$  we have a unique supporting line to  $S_1$ , and we can choose  $f^2(\tau)$  in a proper way. In this case the problem is quite easy and one can see its geometric interpretation.

The general case (when  $C_1$  has non-smooth points) yields additional difficulties. At a point of non-smoothness we have continuously many supporting lines; hence we cannot give a parametrization depending only on this point of  $C_1$ . However, in [21] Joly gives a suitable parametrization.

2) Let  $A_3$  be the measure of the area enclosed by  $C_3$ . Let  $f^i(\cdot)$  be the parametrization of  $C_i$  ( $i = 1, 2$ ) constructed above. Fix  $\mu \in \mathbb{R}$ . Denote by  $S(\mu)$  and  $A(\mu)$  the set and the area enclosed by the curve  $C(\mu) = \{f^1(\tau) + \mu f^2(\tau) : \tau \in [0, 1)\}$ , respectively. Since for all  $\tau \in [0, 1)$  we have that  $f^2(\tau)$  is a direction vector of the supporting line of the set  $S_1$  at the point  $f^1(\tau)$ , then we have  $S_1 \subset S(\mu)$ . Hence  $A(\mu) \geq A_1$ . Using consequences of Green's formula and properties of the Stieltjes integral, we have

$$\int_{\tau \in [0, 1)} f_1^1 df_2^1 \leq \int_{\tau \in [0, 1)} (f_1^1(\tau) + \mu f_1^2(\tau)) d(f_2^1(\tau) + \mu f_2^2(\tau)).$$

Therefore, for all  $\mu \in \mathbb{R}$  the following inequality holds:

$$\mu^2 \int_{\tau \in [0, 1)} f_1^2 df_2^2 + \mu \left( \int_{\tau \in [0, 1)} f_1^1 df_2^2 + \int_{\tau \in [0, 1)} f_1^2 df_2^1 \right) \geq 0.$$

This implies that

$$\left( \int_{\tau \in [0, 1)} f_1^1 df_2^2 + \int_{\tau \in [0, 1)} f_1^2 df_2^1 \right) = 0.$$

So we have

$$A_3 = A(1) = \int_{\tau \in [0,1)} (f_1^1(\tau) + f_1^2(\tau)) d(f_2^1(\tau) + f_2^2(\tau)) = \int_{\tau \in [0,1)} f_1^1 df_2^1 + \int_{\tau \in [0,1)} f_1^2 df_2^2 = A_1 + A_2.$$

□

**Theorem 2.** *Let  $X$  be an arbitrary Banach space. Then the following inequalities hold:*

$$(15) \quad \zeta_X^-(\varepsilon) \leq \zeta_H^-(\varepsilon) = \sqrt{1 + \varepsilon^2} = \zeta_H^+(\varepsilon) \leq \zeta_X^+(\varepsilon).$$

**Proof.**

Again it is sufficient to prove the theorem in the two-dimensional case. Applying Lemma 5 for  $C_1 = \partial \mathfrak{B}_1(o)$ ,  $C_2 = \partial \mathfrak{B}_\varepsilon(o)$  and using continuity arguments we obtain (15).

□

**Remark 4.** *In [21] inequality (15) was proved for the subcase  $\varepsilon = 1$ .*

## 6. SOME NOTES ABOUT MONOTONICITY PROPERTIES OF THE DUAL MAPPING

The notion of monotone operator is well-known and has a lot of applications and useful generalizations. Let us recall some related notions and, based on them, explain their relations to the geometry of the unit sphere.

Let  $X$  be a Banach space,  $T : X \rightarrow X^*$  a point-to-set operator, and  $G(T)$  its graph. Suppose that the following inequality holds:

$$(16) \quad \langle p_x - p_y, x - y \rangle \geq \alpha \|x - y\|^2 \quad \text{for all } (x, p_x), (y, p_y) \in G(T).$$

If

- (1)  $\alpha = 0$ , then  $T$  is a monotone operator. For example, the subdifferential of a convex function is a monotone operator.
- (2)  $\alpha > 0$ , then  $T$  is a strongly monotone operator. For example, the subdifferential of a strongly convex function on a Hilbert space is a strongly monotone operator.
- (3)  $\alpha < 0$ , then  $T$  is a hypomonotone operator. For example, the subdifferential of a prox-regular function on a Hilbert space is a hypomonotone operator (see [19]).

Inequality (16) is often called the *variational inequality*. Usually, the operator  $T$  is a derivative or subderivative of a convex function. So we can speak about the variational inequality for a convex function.

As usual in convex analysis, we can reformulate inequality (16) for convex (or prox-regular) sets and their normal cone (or Frechet normal cone) (see [24]), in this case  $T(x)$  is a intersection of the  $\partial \mathfrak{B}_1^*(o)$  and the normal cone to the set at point  $x$ . In a Hilbert space there are some characterizations of strongly convex and prox-regular functions (or strongly convex and prox-regular sets) via the variational inequality (see [9], [25] and [24]).

But in a Banach space the situation is much more complicated and it is getting obvious that the right-hand side of the variational inequality cannot always be a quadratic function. So, in many applications we have to substitute  $\alpha \|x - y\|^2$  in (16) by some proper convex function  $\alpha(\|x - y\|)$ .

For example, what can we say about the most simple convex function in a Banach space – its norm (in this case  $T$  is a dual mapping)? Even in a Hilbert space, for arbitrary  $x, y$

we can only put zero in the right-side of the variational inequality. Nevertheless, there exist variational inequalities for norms depending on  $\|x\|$ ,  $\|y\|$ , and  $\|x - y\|$ . For example, in [27] characterizations of uniformly smooth and uniformly convex Banach spaces were given in terms of monotonicity properties of the dual mapping.

In this paragraph we investigate monotonicity properties of the dual mapping onto the unit sphere. In fact, we study monotonicity properties of the convex function on its Lebesgue level. Hence this results can be generalized to an arbitrary convex function.

We are interested in asymptotically tight lower and upper bounds for the  $\langle p_1 - p_2, x_1 - x_2 \rangle$ , where  $x_1, x_2 \in \partial\mathfrak{B}_1(o)$ ,  $p_1 \in J_1(x_1)$ ,  $p_2 \in J_1(x_2)$ . For the sake of convenience we introduce new moduli:

$$\gamma_X^+(\varepsilon) = \sup\{\langle p_1 - p_2, x_1 - x_2 \rangle : x_1, x_2 \in \partial\mathfrak{B}_1(o), \|x_1 - x_2\| = \varepsilon, p_1 \in J_1(x_1), p_2 \in J_1(x_2)\}$$

and

$$\gamma_X^-(\varepsilon) = \inf\{\langle p_1 - p_2, x_1 - x_2 \rangle : x_1, x_2 \in \partial\mathfrak{B}_1(o), \|x_1 - x_2\| = \varepsilon, p_1 \in J_1(x_1), p_2 \in J_1(x_2)\}$$

for each  $\varepsilon \in [0, 2]$ .

**Lemma 6.** *Let  $X$  be an arbitrary Banach space. Then the functions  $\gamma_X^+(\cdot)$  and  $\gamma_X^-(\cdot)$  are monotonically increasing functions on  $[0, 2]$ .*

**Proof.**

In the notation of Fig. 1, let  $z_1, z_2$  be points in the arc  $-xyx$  of the unit circle such that  $z_1$  belongs to the arc  $xz_2$  (here and in the sequel all arcs lie in the plane of  $xoy$ ). Let  $p \in J_1(x)$ ,  $q_1 \in J_1(z_1)$ ,  $q_2 \in J_1(z_2)$ . It is worth mentioning that  $\|xz_1\| \leq \|xz_2\|$  (see [1], Lemma 1). So, to prove our Lemma it is sufficient to show that

$$(17) \quad \langle p - q_1, x - z_1 \rangle \leq \langle p - q_2, x - z_2 \rangle$$

From the convexity of the unit ball we have that  $\langle p, x - z_1 \rangle \leq \langle p, x - z_2 \rangle$ . To prove inequality (17), let us show that  $\langle q_1, z_1 - x \rangle \leq \langle q_2, z_2 - x \rangle$ .

We can assume that  $X$  is the plane of  $xoy$ . By definition, put  $l = \{a \in X : \langle a, p \rangle = 1\}$ ,  $l_1 = \{a \in X : \langle a, q_1 \rangle = 1\}$ ,  $l_2 = \{a \in X : \langle a, q_2 \rangle = 1\}$ , and  $H^+ = \{p \in X : \langle a, p \rangle \geq 1\}$ .

The first case: let  $z_2$  be in the arc  $xy$  of the unit circle (see Fig. 3). All three cases  $l = l_1$ ,  $l = l_2$  or  $l_1 = l_2$  are trivial. Let  $l \cap l_1 = \{b_1\}$ ,  $l \cap l_2 = \{b_2\}$ . Again, all three cases  $x = b_1$ ,  $x = b_2$  or  $b_1 = b_2$  are trivial. By convexity arguments,  $b_1$  belongs to the relative interior of the segment  $xb_2$  and  $l_1 \cap l_2 \notin H^+$ . Hence  $l_1$  separates point  $x$  and the ray  $l_2 \cap H^+$  in the half-plane  $H^+$ . Let  $x_2$  be a projection of the point  $x$  onto  $l_2$  (in the non-strictly convex case we choose  $x_2$  such that  $xx_2 \parallel oz_2$ ). Then the segment  $xx_2$  is parallel to  $oz_2$ , and therefore  $xx_2 \subset H^+$ . Now we can say that the segment  $x_2x$  and the line  $l_1$  have an intersection point; let it be  $x_1$ . Since the values  $\langle q_1, z_1 - x \rangle$  and  $\langle q_2, z_2 - x \rangle$  are equal to the distances from the point  $x$  to the lines  $l_1$  and  $l_2$ , respectively, we have:

$$\langle q_1, z_1 - x \rangle \leq \|xx_1\| < \|xx_2\| = \langle q_2, z_2 - x \rangle.$$

The second case: let  $z_2$  be in the arc  $-xy$  of the unit circle. We can assume that  $z_1$  lies on the arc  $-xy$  of the unit circle, too (if  $z_1$  lies on the arc  $xy$  of the unit circle, by the first case we can substitute  $z_1$  to  $y$ ). We have that  $\langle -q_i, -z_i - x \rangle = 2 - \langle q_i, z_i - x \rangle$  for  $i = 1, 2$ . Therefore, applying the first case to the points  $-z_1, -z_2, x$  and to the functionals  $p, -q_1, -q_2$ , we have proved the second case.

□

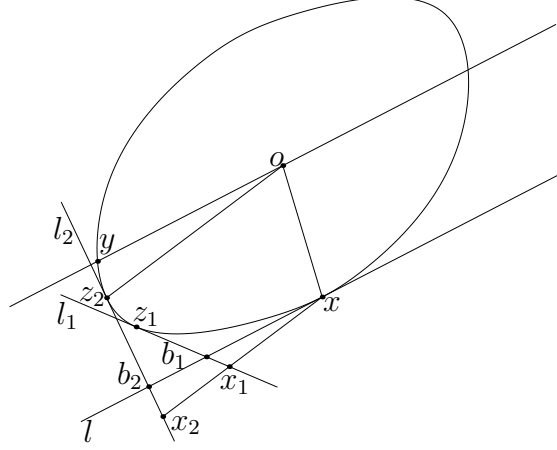


FIGURE 3. Illustration of the proof of Lemma 6

**Remark 5.** It is worth mentioning that in the first case of Lemma 6 the lines  $l_1$  and  $ox$  can have no common point in  $H^+$ .

**Remark 6.** Using Lemma 6, we can modify the definitions of  $\gamma_X^+(\cdot)$  and  $\gamma_X^-(\cdot)$  by

$$\gamma_X^+(\varepsilon) = \sup\{\langle p_1 - p_2, x_1 - x_2 \rangle : x_1, x_2 \in \partial \mathfrak{B}_1(o), \|x_1 - x_2\| \leq \varepsilon, p_1 \in J_1(x_1), p_2 \in J_1(x_2)\}$$

and

$$\gamma_X^-(\varepsilon) = \inf\{\langle p_1 - p_2, x_1 - x_2 \rangle : x_1, x_2 \in \partial \mathfrak{B}_1(o), \|x_1 - x_2\| \geq \varepsilon, p_1 \in J_1(x_1), p_2 \in J_1(x_2)\}$$

for each  $\varepsilon \in [0, 2]$ .

**Lemma 7.** Let  $X$  be an arbitrary Banach space. Then the following inequalities hold:

$$(18) \quad \varphi_X^+(\varepsilon) \leq \gamma_X^+(\varepsilon) \leq 2\varphi_X^+(\varepsilon) \quad \text{for } \varepsilon \in [0, 2],$$

$$(19) \quad 2\varphi_X^-\left(\frac{\varepsilon}{4}\right) \leq \gamma_X^-\left(\frac{\varepsilon}{4}\right) \leq \varphi_X^-(\varepsilon) \quad \text{for } \varepsilon \in [0, 1].$$

**Proof.**

All inequalities, except for the right-hand side of (19), are obvious.

Let us prove that  $\gamma_X^-\left(\frac{\varepsilon}{4}\right) \leq 2\varphi_X^-(\varepsilon)$ . It is sufficient to prove the lemma in the two-dimensional case. In this case and in the notation of Fig. 1 we can put  $\|zx\| = \varepsilon$  and  $\|y_1z\| = \varphi_X^-(\varepsilon)$ . Let  $y_b$  be a bisecting point of the segment  $xy_1$ . Denote by  $z_b$  a point from the unit sphere such that  $z_by_b \parallel ox$  and  $z_by_b \cap \mathfrak{B}_1(o) = \{z_b\}$ . Let  $p_b \in J_1(z_b)$ . Denote by  $l_b$  the line  $\{a \in X : \langle p_b, a \rangle = 1\}$ . By convexity the line  $l_b$  intersects the segment  $zy_1$ , and we denote the intersection point as  $a_1$ . By definition put  $\{a_2\} = l_1 \cap \{\tau x : \tau \in \mathbb{R}\}$ . From the trapezoid  $a_2x a_1 y_1$  we have that

$$(20) \quad \|y_b z_b\| + \|x a_2\| \leq \|y_1 a_1\| \leq \|z y_1\| = \varphi_X^-(\varepsilon).$$

Since  $\langle p_b, z_b - x \rangle$  equals the distance from the point  $x$  to the line  $l_b$ , we have that  $\langle p_b, z_b - x \rangle \leq \|xa_2\|$ . From here, since  $\langle p, x - z_b \rangle = \|y_b z_b\|$ , and from inequality (20) we obtain

$$\langle p - p_b, x - z_b \rangle \leq \varphi_X^-(\varepsilon).$$

From Lemma 6 it is sufficient to show that  $\|xz_b\| \geq \frac{\varepsilon}{4}$ . By definition put  $\{z'\} = y_b z_b \cap xz$ . Obviously, we have that

$$\|xz_b\| \geq \|xz'\| - \|z'z_b\| \geq \|xz'\| - \|z'y_b\| = \frac{\varepsilon - \varphi_X^-(\varepsilon)}{2}.$$

Using Theorem 1, we see that

$$\|xz_b\| \geq \frac{\varepsilon}{2} - \frac{\varepsilon^2}{4} \geq \frac{\varepsilon}{4}.$$

□

**Corollary 3.** *Let  $X$  be an arbitrary Banach space. Then  $\gamma_X^+(\varepsilon) \asymp \rho_X(\varepsilon)$  and  $\gamma_X^-(\varepsilon) \asymp \delta_X(\varepsilon)$  as  $\varepsilon \rightarrow 0$  and for  $\varepsilon \in [0, \frac{1}{2}]$  the following inequalities hold:*

$$\rho_X\left(\frac{\varepsilon}{4}\right) \leq \gamma_X^+(\varepsilon) \leq 2\rho_X(4\varepsilon) \quad \text{and}$$

$$2\delta_X\left(\frac{\varepsilon}{4}\right) \leq \gamma_X^-\left(\frac{\varepsilon}{4}\right) \leq \delta_X(\varepsilon).$$

**Remark 7.** *Combining results from [27] for some constant  $c_1, c_2, c_3, c_4$  (depending on  $X$ ) one can get the following inequality:*

$$c_1\rho_X(c_2\varepsilon) \geq \gamma_X^+(\varepsilon) \geq \gamma_X^-(\varepsilon) \geq c_3\delta_X(c_4\varepsilon).$$

## 7. SOME OPEN QUESTIONS

Although there are no difficulties to prove an analogue of the Day-Nordlander theorem for the moduli  $\gamma_X^+(\cdot)$  and  $\gamma_X^-(\cdot)$  moduli in the infinite-dimensional case using Dvoretzky's theorem (see [15]), we have no proof for the following conjecture in the finite-dimensional case:

**Conjecture 1.** *Let  $X$  be an arbitrary Banach space. Then the following inequalities hold:*

$$(21) \quad \gamma_X^-(\varepsilon) \leq \gamma_H^-(\varepsilon) = \varepsilon^2 = \gamma_H^+(\varepsilon) \leq \gamma_X^+(\varepsilon).$$

All moduli mentioned above characterize certain geometrical properties of the unit ball. Obviously, the geometry of the unit ball totally describes the geometry of the unit ball in the dual space. Nevertheless, we know a few results about coincidences of values of some moduli or other characteristics of a Banach space and its dual space. We are interested in properties of the dual mapping (i.e.,  $x \rightarrow J_1(x)$ ). The following conjecture seems to be very essential. By definition, put

$$d_X^-(\varepsilon) = \inf\{\|p_1 - p_2\| \mid p_1 \in J_1(x_1), p_2 \in J_1(x_2), \|x_1 - x_2\| = \varepsilon, x_1, x_2 \in \partial\mathfrak{B}_1(o)\}$$

and

$$d_X^+(\varepsilon) = \sup\{\|p_1 - p_2\| \mid p_1 \in J_1(x_1), p_2 \in J_1(x_2), \|x_1 - x_2\| = \varepsilon, x_1, x_2 \in \partial\mathfrak{B}_1(o)\}.$$

**Conjecture 2.** *Let  $X$  be an arbitrary Banach space. Then the following inequalities hold:*

$$(22) \quad d_X^-(\varepsilon) \leq d_H^-(\varepsilon) = \varepsilon = d_H^+(\varepsilon) \leq d_X^+(\varepsilon).$$

It is well-known that the equality  $\delta_X(\varepsilon) = \delta_H(\varepsilon)$  for  $\varepsilon \in [0, 2)$  implies that  $X$  is an inner product space (see [13]). There exist such results for some other moduli (See [1] and [4]). We are interested in the following question:

**Question 1.** *For what modulus  $f_X(\cdot)$  ( $f_X(\cdot) = \varphi_X^-(\cdot), \varphi_X^+(\cdot), \zeta_X^-(\cdot), \zeta_X^+(\cdot), \lambda_X^-(\cdot), \lambda_X^+(\cdot)$ ) does the equality  $f_X(\varepsilon) = f_H(\varepsilon)$ , holding for all  $\varepsilon$  in the domain of the function  $f_X(\cdot)$  (or even for fixed  $\varepsilon$ ), imply that  $X$  is an inner product space?*

The definitions of the moduli  $\zeta_X^-(\cdot) - 1$  and  $\zeta_X^+(\cdot) - 1$  are similar to the definitions of Milman's moduli, which were introduced in [22] as

$$\beta_X^-(\varepsilon) = \inf_{x, y \in \partial \mathfrak{B}_1(o)} \{\max\{\|x + \varepsilon y\|, \|x - \varepsilon y\|\} - 1\}$$

and

$$\beta_X^+(\varepsilon) = \sup_{x, y \in \partial \mathfrak{B}_1(o)} \{\min\{\|x + \varepsilon y\|, \|x - \varepsilon y\|\} - 1\}.$$

We think that in the definitions of Milman's moduli it is sufficient to take only  $y \lrcorner x$ . Hence we get

**Conjecture 3.** *Let  $X$  be an arbitrary Banach space. Then for positive  $\varepsilon$  we have*

$$\zeta_X^-(\varepsilon) - 1 = \beta_X^-(\varepsilon) \text{ and } \zeta_X^+(\varepsilon) - 1 = \beta_X^+(\varepsilon).$$

## REFERENCES

- [1] J. Alonso and C. Benitez. Some characteristic and non-characteristic properties of inner product spaces. *J. Approx. Theory*, 55:318–325, 1988.
- [2] J. Alonso, H. Martini, and M. Spirova. Minimal enclosing discs, circumcircles, and circumcenters in normed planes, II. *Comput. Geom.*, 45(7):350–369, 2012.
- [3] J. Alonso, H. Martini, and S. Wu. On Birkhoff orthogonality and isosceles orthogonality in normed linear spaces. *Aequationes Math.*, 83(1-2):153B–189, 2012.
- [4] D. Amir. *Characterizations of Inner Product Spaces*. Basel: Birkhäuser Verlag, 1986.
- [5] J. Banaś. On moduli of smoothness of Banach spaces. *Bull. Pol. Acad. Sci., Math.*, 34:287–293, 1986.
- [6] J. Banaś and K. Frączek. Deformation of Banach spaces. *Comment. Math. Univ. Carolinae*, 34:47–53, 1993.
- [7] J. Banaś, A. Hajnosz, and S. Wędrychowicz. On convexity and smoothness of Banach space. *Commentationes Mathematicae Universitatis Carolinae*, 31(3):445–452, 1990.
- [8] M. Baronti and P. Papini. Convexity, smoothness and moduli. *Nonlinear Analysis: Theory, Methods & Applications*, 70(6):2457–2465, 2009.
- [9] F. H. Clarke, R. J. Stern, and P. R. Wolenski. Proximal smoothness and lower- $c^2$  property. *J. Convex Anal.*, 2(1):117–144, 1995.
- [10] J. A. Clarkson. Uniformly convex spaces. *Trans. Amer. Math. Soc.*, 40:396–414, 1936.
- [11] I. Šerb. A Day-Nordlander theorem for the tangential modulus of a normed space. *J. Math. Anal. Appl.*, 209:381–391, 1997.
- [12] M. M. Day. Uniform convexity in factor and conjugate spaces. *Ann. of Math.*, 45:375–385, 1944.
- [13] M. M. Day. Some characterisations of inner product spaces. *Trans. Amer. Math. Soc.*, 62:320–337, 1947.
- [14] J. Diestel. *Geometry of Banach Spaces - Selected Topics*, volume 485. Springer-Verlag Berlin Heidelberg, 1975.
- [15] A. Dvoretzky. Some results on convex bodies and Banach spaces. *Proc. Internat. Sympos. Linear Spaces*, pages 123–160, 1961.
- [16] A. Guirao and P. Hajek. On the moduli of convexity. *Proc. Amer. Math. Soc.*, 135(10):3233–3240, 2007.
- [17] A. J. Guirao, M. Ivanov, and S. Lajara. On moduli of smoothness and squareness. *J. Convex Anal.*, 17:441–449, 2010.

- [18] C. He and Y. Cui. Some properties concerning Milman's moduli. *J. Math. Anal. Appl.*, 329:1260–1272, 2007.
- [19] G. E. Ivanov. *Weakly Convex Sets and Functions. Theory and Applications. (in Russian)*. Moscow, 2006.
- [20] G. M. Ivanov. Modulus of supporting convexity and supporting smoothness. *Eurasian Math. J.*, 6(1):26–40, 2015.
- [21] J. Joly. Characterisations d'espaces hilbertiens au moyen de la constante rectangle. *J. Approx. Theory*, 2:301–311, 1969.
- [22] V. D. Milman. Geometric theory of Banach spaces. Part II. Geometry of the unit sphere. *Uspechi Mat. Nauk*, 26(6):73–149, 1971.
- [23] G. Nordlander. The modulus of convexity in normed linear space. *Ark Mat.*, 4(1):15–17, 1960.
- [24] R. Poliquin, R. Rockafellar, and L. Thibault. Local differentiability of distance functions. *Trans. Amer. Math. Soc.*, 352(11):5231–5249, 2000.
- [25] R. Poliquin and R. Rockafellar. Prox-regular functions in variational analysis. *Trans. Amer. Math. Soc.*, 368:1805–1838, 1996.
- [26] M. Rio and C. Benitez. The rectangular constant for two-dimensional spaces. *J. Approx. Theory*, 19:15–21, 1977.
- [27] Z.-B. Xu and G. F. Roach. Characteristic inequalities of uniformly convex and uniformly smooth Banach spaces. *J. Math. Anal. Appl.*, 157:189–210, 1991.